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# Approximate Solutions of the Membrane Flutter Problem

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The linearised dynamic equation associated with two-dimensional membrane flutter in supersonic flow may be written (using the "static" aerodynamic approximation) as

$$\frac{\partial^2 Y}{\partial x^2} - \alpha \frac{\partial Y}{\partial x} - \gamma \frac{\partial^2 Y}{\partial t^2} = 0 \tag{1}$$

where $\alpha$ and $\gamma$ are constants.

Separation of variables by writing $Y = y(x)e^{i\omega t}$ gives

$$\frac{d^2 y}{dx^2} - \alpha \frac{dy}{dx} + \lambda y = 0 \tag{2}$$

It is now necessary to find eigenvalues $\lambda = \gamma\omega^2$ for a nontrivial solution to Eq. (2) subject to the boundary conditions $y(0) = y(1) = 0$. Exact solution leads to the result

$$\lambda = n^2\pi^2 + (\alpha^2/4) \tag{3}$$

The question of approximate solution of this problem using the Galerkin method has raised doubts about the validity of this method with a non-self-adjoint second-order equation. It has been shown that the second and third terms in an approximating series lead to critical values of $\alpha$ at which two eigenvalues coalesce to form complex conjugates. The exact solution shows this flutter condition is spurious. However, the work of Keldysh[1] and Petrov[2] indicates that the exact eigenvalues of Eq. (2) can be obtained as the limit of a sequence of approximate eigenvalues (from the Galerkin method) formed by increasing the number of terms in the approximating series. It might be argued then that an examination of the effect of increasing the number of terms in the series on the critical value of $\alpha$ may show the spurious nature of the flutter boundary.

A Galerkin solution using the approximating series

$$y = \sum_{n=1}^{\infty} A_n \sin n\pi x \tag{4}$$

in Eq. (2) leads to a matrix problem in which it is necessary to find the eigenvalues $\lambda$ of the matrix $\{a_{mn}\}$ where

$$a_{nn} = n^2\pi^2$$

$$a_{mn} = \frac{2mn}{(m^2 - n^2)} \alpha [1 - (-)^{m+n}] \qquad m \neq n \tag{5}$$

Since this matrix is not symmetric, it is quite possible that for some values of $\alpha$ the eigenvalues have nonzero imaginary parts. The eigenvalues of increasing order matrices have been evaluated and the critical value of $\alpha$ determined by the criterion that at flutter two of the eigenvalues will coalesce to form a complex conjugate pair.

Table 1 gives the number of modes and the critical value of $\alpha$ for each. If a greater number of terms were taken, $\alpha_{crit}$ would form a more obviously diverging sequence. An important feature of the solutions is that of instability occurring when the two highest eigenvalues coalesce. However, since only the lower eigenvalues have been determined with any accuracy, it is obvious at any stage that more terms must be taken to check the instability; as Bisplinghoff[3] suggests, an infinite number of terms are in fact needed. This leads to the conclusion that, although a finite $\alpha_{crit}$ has been obtained, it has no meaning because of the manner in which the instability occurs.

It is well known that Galerkin's method applied to a non-self-adjoint differential equation need have no relationship to the Ritz method applied to a variational problem. However, in the case of a second-order differential equation, it is always possible to introduce a multiplying factor that will convert the equation to the self-adjoint form. For Eq. (2), it is $e^{-\alpha x}$, giving the self-adjoint equation,

$$d/dx[e^{-\alpha x}(dy/dx)] + \lambda e^{-\alpha x} y = 0 \tag{6}$$

(a particular case of the homogeneous Sturm-Liouville differential equation).

All second-order self-adjoint differential equations may be regarded as Euler equations arising from variational problems with homogeneous quadratic integrands. In this case, Eq. (6) is the Euler equation for the minimization of the integral

$$\int_0^1 e^{-\alpha x} \left\{ \left(\frac{dy}{dx}\right)^2 - \lambda y^2 \right\} dx \tag{7}$$

In general, Galerkin's method applied to a self-adjoint equation leads to an eigenvalue problem involving symmetric matrices. When this technique is applied to Eq. (6) with the approximating series used previously, it leads to an eigenvalue problem of the form

$$|B - \lambda C| = 0 \tag{8}$$

where $B$ and $C$ are symmetric matrices whose elements are given as

$$b_{mn} = \frac{[1 - (-)^{m+n}e^{-\alpha}][(m^2 + n^2)\pi^2 + \alpha^2]}{[(m - n)^2\pi^2 + \alpha^2][(m + n)^2\pi^2 + \alpha^2]}$$

$$c_{mn} = \frac{[1 - (-)^{m+n}e^{-\alpha}]}{[(m - n)^2\pi^2 + \alpha^2][(m + n)^2\pi^2 + \alpha^2]}$$

If the matrix $C$ is positive definite, the eigenvalues of Eq. (8) are real, and Galerkin's method would lead directly to the correct result. Even a two-term solution gives

$$2\lambda = 5\pi^2 + \alpha^2 \pm \frac{3\pi^2(1 - e^{-\alpha})(\pi^2 + \alpha^2)(9\pi^2 + \alpha^2)}{\{(1 - e^{-\alpha})^2(\pi^2 + \alpha^2)^2(9\pi^2 + \alpha^2)^2 - \alpha^4(1 + e^{-\alpha})^2(4\pi^2 + \alpha^2)(16\pi^2 + \alpha^2)\}^{1/2}}$$

The denominator of the last term on the right-hand side is always real, hence a two-term Galerkin analysis of the self-adjoint equation indicates no instability.

Two conclusions may be drawn from this study: 1) the Galerkin method is applicable to the case of the supersonic membrane flutter problem and yields results in agreement with the exact solution when these results are interpreted correctly; any approximate solution should involve a convergence examination; and 2) if the dynamic equation is made self-adjoint, a two-term solution gives the correct answer without it being necessary to investigate the problem more deeply.

Finally it should be noted that it is rarely necessary to solve (exactly or approximately) any boundary-value prob-

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**Table 1 Variation in $\alpha_{crit}$**

| Number of terms | $\alpha_{crit}$ (to 2 significant figures) |
|---|---|
| 2 | 5.5 |
| 5 | 5.6 |
| 10 | 5.7 |
| 15 | 5.8 |

lem originating from a second-order linear differential equation having homogeneous boundary conditions. Theoretical treatment indicates that a condition for the eigenvalues of such a system to be always real is that the quotient of the coefficients of the second derivative and the "$\lambda y$" term be positive in the interval.[4] For Eq. (2) the coefficients are constants and the result follows trivially.

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# Normal Shock Location of Underexpanded Gas-Particle Jets

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### Nomenclature

$\eta$ = $r/R(x)$
$x$ = axial distance from the nozzle exit plane
$r$ = radial distance from axis
$R(x)$ = radius of the outer edge of the gas cloud or plume
$R_0$ = radius of the nozzle exit plane
$v$ = velocity normal to the axis
$\rho$ = gas density
$\rho_0$ = averaged value of $\rho$ at $x = 0$
$a_0$ = sonic speed at $x = 0$
$B(\gamma)$ = $1/(\gamma - 1)$
$D(\gamma)$ = an unknown function

### Introduction

An important consideration in underexpanded rocket nozzle plumes is the location of the normal shock, since the shock structure affects the radiation of the plume. This note makes use of a simple physical model to show how solid particles affect the normal shock location.

The model is based on the assumption that the ambient pressure is much smaller than the jet exit pressure, producing a greatly underexpanded jet. Under these conditions, the effect of the ambient pressure on the flow near the plume axis may be neglected. The flow parameters near the axis are calculated by the analogy between the unsteady expansion of a cylindrical gas cloud and the steady hypersonic flow from a nozzle. Hence, the model applies only to nozzles that have a sufficiently high exit Mach number.

The model also uses the assumption of no particle lag in the supersonic portion of the plume, although it treats both no-lag and complete-lag flows behind the normal shock. Except for the recognition of particle drag, viscous effects are neglected throughout the analysis.

The normal shock location is based on the method given by Adamson[1] and by Adamson and Nicholls,[2] since it is simpler to apply than the methods of D'Attorre and Harshbarger[3] and Eastman and Radtke.[4]

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### Expansion of the Rocket Plume

The hypersonic blast or expanding gas-cloud analogy used here has been pointed out both by Mirels and Mullen[5] and by Greifinger and Cole.[6] The essential point is that the flow problem parallel to the rocket plume axis may be decoupled mathematically from the flow normal to the axis by neglecting terms of order $\delta^2$ as compared with 1, where $\delta$ is a characteristic slope of the plume streamlines. This will be true if the nozzle-exit plane Mach number $M_0$ is such that $M_0^2 \gg 1$, provided that the gas specific heat ratio $\gamma$ is not nearly equal to 1. Then, the radial flow problem is just that of the gas cloud.

For the expansion of a cylindrical gas cloud, both Sedov[7] and Keller[8] show that, after a long time, a self-similar solution exists in the form of

$$\rho/\rho_0 = D(\gamma)(R_0M_0/x)^2(1 - \eta^2)^{B(\gamma)} \quad (1)$$

$$v/a_0 = \eta(R_0M_0/x)(R/R_0) \quad (2)$$

where the dimensionless time $t$ has been replaced by the dowhstream distance $(x/R_0M_0)$ and where $R$ is given for large values of $x/(M_0R_0)$ as

$$R(x) = [2^{1/2}/(\gamma - 1)](x/M_0R_0)R_1 \quad (3)$$

These equations are valid for $R^{2(\gamma-1)}$ large compared to $R_0^{2(\gamma-1)}$; $R_1$ will be defined later.

At $x = 0$, the rocket nozzle has a density distribution that is, in general, different from $(1 - \eta^2)^{1/(\gamma-1)}$. Hence, the question remains of whether the distribution will tend to approach the form $(1 - \eta^2)^{1/(\gamma-1)}$ for large values of $(x/M_0R_0)$. Mirels and Mullen[5] settle the question by equating the mass-flow rate past $x = 0$ to the mass-flow rate at large values of $(x/M_0R_0)$, and by doing the same for energy-flow rate. They also assume the condition that $R_1 = R_0$, where $R_1$ is the radius of the self-similar density distribution at $x = 0$ that is equivalent to the real rocket nozzle density distribution at $x = 0$. These three conditions determine the three unknowns $D(\gamma)$, $B(\gamma)$, and $R_1$. This leads to $B(\gamma) = 2/(\gamma - 1)$, although $1/(\gamma - 1)$ is the only self-similar exponent that satisfies the equations of motion.[7, 8] Of course, it has not been shown that the plume would be stable in or tend toward the self-similar form, but in the absence of any evidence to the contrary, it will be assumed that the plume approaches this form, so that $B(\gamma) = 1/(\gamma - 1)$. Then we may use the mass and energy conservation equations to calculate $D(\gamma)$ and the ratio $(R_1/R_0)$.

Mass conservation gives

$$\rho_0u_0\pi R_0^2 = R^2 \int_0^1 \rho u_m 2\pi\eta dy \quad (4)$$

and energy conservation gives

$$\left(\frac{\rho_0a_0^2}{\gamma - 1}\right)\pi R_0^2 = R^2 \int_0^1 \rho k^2 \frac{v^2}{2} 2\pi\eta dy \quad (5)$$

where $u_0$ = axial velocity at $x = 0$, $u_m$ = axial velocity at large $x$, and $k = u_m/u_0$. The distinction between $u_0$ and $u_m$ is added for the purpose of extending the results to cases where $M_0$ is not strictly hypersonic, that is, to cases where the gas experiences a small but significant axial acceleration just downstream of the exit plane. This, of course, implies some coupling between the axial and the radial-flow problems, which is neglected here. In Eq. (4), the mass flux at $x = 0$ is equated to the mass flux at large values of $x/R_0M_0$.